

# Noether-Lefschetz locus and a special case of the variational Hodge conjecture: Using elementary techniques

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## Abstract

Fix integers  $n \geq 1$  and  $d$  such that  $nd > 2n + 2$ . The Noether-Lefschetz locus  $\text{NL}_{d,n}$  parametrizes smooth projective hypersurfaces in  $\mathbb{P}^{2n+1}$  such that  $H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q}) \neq \mathbb{Q}$ . An irreducible component of the Noether-Lefschetz locus is locally a Hodge locus. One question is to ask under what choice of a Hodge class  $\gamma \in H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q})$  does the variational Hodge conjecture hold true? In this article we use methods coming from commutative algebra and Hodge theory to give an affirmative answer in the case  $\gamma$  is the class of a complete intersection subscheme in  $X$  of codimension  $n$ . Another problem studied in this article is: In the case  $n = 1$  when is an irreducible component of the Noether-Lefschetz locus nonreduced? Using the theory of infinitesimal variation of Hodge structures of hypersurfaces in  $\mathbb{P}^3$ , we determine all non-reduced components with codimension less than or equal to  $3d$  for  $d \gg 0$ . Here again our primary tool is commutative algebra.

**Notation 0.1.** Throughout this article,  $X$  will denote a smooth hypersurface in  $\mathbb{P}^{2n+1}$ . Denote by  $H^{n,n}(X, \mathbb{Q})$  the intersection  $H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q})$  and  $H_X$  the very ample line bundle on  $X$ .

## 1 Introduction

It was first stated by M. Noether and later proved by S. Lefschetz that for a general smooth surface  $X$  in  $\mathbb{P}^3$ , the rank of the Néron-Severi group, denoted  $\text{NS}(X)$  is of rank 1. We can then

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define the *Noether-Lefschetz locus*, denoted  $\text{NL}_{d,1}$ , to be the space of smooth degree  $d$  surfaces in  $\mathbb{P}^3$  with Picard rank greater than 1. Using Lefschetz  $(1,1)$ -theorem, one can see that  $\text{NL}_{d,1}$  is the space of smooth degree  $d$  surfaces  $X$  such that  $H^{1,1}(X, \mathbb{Q}) \neq \mathbb{Q}$ . Similarly, we can define *higher Noether-Lefschetz locus* as follows: Let  $n > 1$  and  $d$  another integer such that  $nd > 2n+2$ . Denote by  $\text{NL}_{d,n}$  the space of smooth degree  $d$  hypersurfaces  $X$  in  $\mathbb{P}^{2n+1}$  such that  $H^{n,n}(X, \mathbb{Q}) \neq \mathbb{Q}$ . The orbit of the action of the monodromy group on a rational class is finite (see [CDK95]). Consequently,  $\text{NL}_{d,n}$  is an uncountable union of algebraic varieties (see [Voi03, §3.3] for more details).

Let  $L$  be an irreducible component of  $\text{NL}_{d,n}$ . Then  $L$  can be locally studied as the Hodge locus corresponding to a Hodge class. In particular, take  $X \in L$ , general and consider the space of all smooth degree  $d$  hypersurfaces in  $\mathbb{P}^{2n+1}$ , denoted  $U_{d,n}$ . For  $X \in L$ , general, there exists  $\gamma \in H^{n,n}(X, \mathbb{Q})$  and an open (analytic) simply connected set  $U$  in  $U_{d,n}$  containing  $X$  such that  $L \cap U$  is the Hodge locus corresponding to  $\gamma$ , denoted  $\text{NL}_{d,n}(\gamma)$  (see [Voi02, §5.3] for more details). Before we state the first main result in this article, we fix some notations. Given a Hilbert polynomial  $P$ , of a subscheme  $Z$ , in  $\mathbb{P}^{2n+1}$ , denote by  $H_P$  the corresponding Hilbert scheme. Denote by  $Q_d$  the Hilbert polynomial of a degree  $d$  hypersurface in  $\mathbb{P}^{2n+1}$ . The flag Hilbert scheme  $H_{P,Q_d}$  parametrizes all pairs  $(Z, X)$ , where  $Z \in H_P$ ,  $X$  is a smooth degree  $d$  hypersurface in  $\mathbb{P}^{2n+1}$  containing  $Z$ . For any  $n \geq 1$  we prove the following theorem which is a special case of the variational Hodge conjecture:

**Theorem 1.1.** Let  $Z$  be a complete intersection subscheme in  $\mathbb{P}^{2n+1}$  of codimension  $n+1$ . Assume that there exists a smooth hypersurface in  $\mathbb{P}^{2n+1}$ , say  $X$ , containing  $Z$ , of degree  $d > \deg(Z)$ . For the cohomology class  $\gamma = a[Z] \in H^{n,n}(X, \mathbb{Q})$ ,  $a \in \mathbb{Q}$ ,  $\gamma$  remains of type  $(n, n)$  if and only if  $\gamma$  remains an algebraic cycle. In particular,  $\overline{\text{NL}_{d,n}(\gamma)}$  (closure taken in  $U_{d,n}$ ) is isomorphic to an irreducible component of  $\text{pr}_2 H_{P,Q_d}$  which parametrizes all degree  $d$  hypersurfaces in  $\mathbb{P}^{2n+1}$  containing a complete intersection subscheme with Hilbert polynomial  $P$ , where  $P$  (resp.  $Q_d$ ) is the Hilbert polynomial of  $Z$  (resp.  $X$ ).

In [Otw03], Otwinowska proves this statement for  $d \gg 0$ . Furthermore, in the case  $n = 1$ , we prove:

**Theorem 1.2.** Let  $d \geq 5$  and  $\gamma$  is a divisor in a smooth degree  $d$  surface of the form  $\sum_{i=1}^r a_i [C_i]$

with  $C_i$  distinct integral curves for all  $i = 1, \dots, r$  and  $d > \sum_{i=1}^r a_i \deg(C_i) + 4$ . Then the following are true:

- (i) If  $r = 1$  and  $\deg(C_1) < 4$  then  $\overline{\text{NL}_{d,1}(\gamma)}$  (closure taken under Zariski topology on  $U_{d,1}$ ) is reduced. In particular,  $\overline{\text{NL}_{d,1}(\gamma)}$  is an irreducible component of  $\text{pr}_2(H_{P,Q_d})$ , the space parametrizing all degree  $d$  surfaces containing a reduced curve with the same Hilbert polynomial as  $C_1$ , which we denote by  $P$ .
- (ii) Suppose that  $r > 1$ . For  $d \gg 0$ , every irreducible component  $L$  of  $\text{NL}_{d,1}$  of codimension at most  $3d$  is locally of the form  $\text{NL}_{d,1}(\gamma)$  with  $\gamma$  as above,  $\deg(C_i) \leq 3$  and  $\overline{\text{NL}_{d,1}(\gamma)}_{\text{red}} = \bigcap_{i=1}^r \overline{\text{NL}_{d,1}([C_i])}_{\text{red}}$ . Moreover,  $\overline{\text{NL}_{d,1}(\gamma)}$  is non-reduced if and only if there exists a pair  $(i, j)$ ,  $i \neq j$  such that  $C_i \cdot C_j \neq 0$ .

## 2 Proof of Theorem 1.1

**Notation 2.1.** Denote by  $S_n^k$  the degree  $k$ -graded piece of  $H^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(k))$ . Define  $S_n := \bigoplus_{k \geq 0} S_n^k$ . Let  $X$  be a smooth degree  $d$  hypersurface in  $\mathbb{P}^{2n+1}$ , defined by an equation  $F$ . Denote by  $J_F$ , the Jacobian ideal of  $F$  generated as an  $S_n$ -module by the partial derivatives of  $F$  with respect to  $\frac{\partial}{\partial X_i}$  for  $i = 1, \dots, 2n+1$ , where  $X_i$  are the coordinates of  $\mathbb{P}^{2n+1}$ . Define,  $R_F := S_n/J_F$ . For  $k \geq 0$ , let  $J_F^k$  (resp.  $R_F^k$ ) symbolize the degree  $k$ -graded piece of  $J_F$  (resp.  $R_F$ ).

**2.2.** We now recall some standard facts about Hodge locus. Let  $X$  be a smooth projective hypersurface in  $\mathbb{P}^{2n+1}$  of degree  $d$ . Recall, there is a natural morphism from  $H^{n,n}(X)$  to  $H^{n,n}(X)_{\text{prim}}$ , where  $H^{n,n}(X)_{\text{prim}}$  denotes the primitive cohomology on  $H^{n,n}(X)$  (see [Voi02, §6.2, 6.3] for more on this topic). Denote by  $\gamma_{\text{prim}}$  the image of  $\gamma$  under this morphism. Using the Lefschetz decomposition theorem, one can see that  $\text{NL}_{d,n}(\gamma)$  coincides with  $\text{NL}_{d,n}(\gamma_{\text{prim}})$  i.e.,  $\gamma$  remains of type  $(n, n)$  if and only if so does  $\gamma_{\text{prim}}$ .

**2.3.** Now,  $K_{\mathbb{P}^{2n+1}} = \mathcal{O}_{\mathbb{P}^{2n+1}}(-2n-2)$ ,  $H^0(K_{\mathbb{P}^{2n+1}}(2n+2)) = H^0(\mathcal{O}_{\mathbb{P}^{2n+1}}) \cong \mathbb{C}$  generated by

$$\Omega := X_0 \dots X_{2n+1} \sum_i (-1)^i \frac{dX_0}{X_0} \wedge \dots \wedge \frac{d\hat{X}_i}{X_i} \wedge \dots \wedge \frac{dX_{2n+1}}{X_{2n+1}},$$

where the  $X_i$  are homogeneous coordinates on  $\mathbb{P}^{2n+1}$ . Recall, for the closed immersion  $j :$

$X \rightarrow \mathbb{P}^{2n+1}$ , denote by  $H^{2n}(X, \mathbb{Q})_{\text{van}}$ , the kernel of the Gysin morphism  $j_*$  from  $H^{2n}(X, \mathbb{Q})$  to  $H^{2n}(\mathbb{P}^{2n+1}, \mathbb{Q})$ . Now, [Voi03, Theorem 6.5] tells us that there is a surjective map,

$$\alpha_{n+1} : H^0(\mathbb{P}^{2n+1}, \mathcal{O}_{\mathbb{P}^{2n+1}}((n+1)d - 2n - 2)) \rightarrow F^{n+1}H^{2n+1}(\mathbb{P}^{2n+1} \setminus X, \mathbb{C}) \cong F^n H^{2n}(X, \mathbb{C})_{\text{van}}$$

which sends a polynomial  $P$  to the residue of the meromorphism form  $P\Omega/F^{n+1}$ , where  $F$  is the defining equation of  $X$  (see [Voi03, §6.1] for more). Finally, [Voi03, Theorem 6.10] implies that  $\alpha_{n+1}$  induces an isomorphism between  $R_F^{(n+1)d-(2n+2)}$  and  $H^{n,n}(X)_{\text{prim}}$ .

**2.4.** We now recall a theorem due to Macaulay which will be used throughout this article. A sequence of homogeneous polynomials  $G_i \in S_n^{d_i}, i = 0, \dots, 2n+1$  with  $d_i > 0$  is said to be *regular* if the  $G_i$  have no common zero. Denote by  $I_G$  the ideal in  $S_n$  generated by the polynomials  $P_i$  for  $i = 0, \dots, 2n+1$ . Denote by  $H_G$  the quotient  $S_n/I_G$  and by  $H_G^i$  the degree  $i$  graded piece in  $H_G$ .

**Theorem 2.5** (Macaulay). Let  $N := \sum_{i=0}^{2n+1} d_i - 2n - 2$ . Then, the rank of  $H_G^N = 1$  and for every integer  $k$ , the pairing,  $H_G^k \times H_G^{N-k} \rightarrow H_G^N$  is perfect.

See [Voi03, Theorem 6.19] for the proof of the statement.

**2.6.** Denote by  $P \in S_n^{(n+1)d-(2n+2)}$  such that  $\alpha_{n+1}(\bar{P}) = \gamma$ . Using [Voi03, Theorem 6.17], we observe that  $T_X \text{NL}_{d,n}(\gamma)$  is isomorphic to the preimage of  $\ker(\bar{P} : R_F^d \rightarrow R_F^{(n+2)d-(2n+2)})$  under the natural quotient morphism from  $S_n^d \rightarrow S_n^d/J_F^d$ .

**2.7.** It is easy to see that for any  $\gamma' \in H^{n,n}(X, \mathbb{Q})$ ,  $\text{NL}_{d,n}(\gamma') = \text{NL}_{d,n}(a'\gamma')$  for any  $a' \in \mathbb{Q}$ , non-zero. For the rest of this section, we assume  $\gamma = [Z]$ , where  $Z$  is as in the statement of the theorem.

**Notation 2.8.** Denote by  $N := (n+1)d - (2n+2)$ . Since  $X$  is smooth, the corresponding Jacobian ideal  $J_F$  can be generated by a regular sequence of  $2n+2$  polynomials  $G_i$  of degree  $d-1$ . Using Theorem 2.5, we see that there exists a perfect pairing  $R_F^k \times R_F^{2N-k} \rightarrow R_F^{2N}$  for all  $k \leq 2N$  and  $R_F^{2N}$  is one dimensional complex vector space. Denote by  $T'_0$ , the subspace of  $R_F^N$  which is the kernel under the multiplication map,  $\cdot P : R_F^N \rightarrow R_F^{2N}$ . Denote by  $T_0$  the preimage of  $T'_0$  in  $S_n^N$  under the natural projection map from  $S_n^N$  to  $R_F^N$ . Define  $T_1$  the subspace

of  $S_n$ , a graded  $S_n$ -module such that for all  $t \geq 0$ , the  $t$ -graded piece of  $T_1$ , denoted  $T_{1,t}$  is the largest subvector space of  $S_n^t$  such that  $T_{1,t} \otimes S_n^{N-t}$  is contained in  $T_0$  for  $t < N$ ,  $T_{1,N} = T_0$  and  $T_{1,N+t} = T_0 \otimes S_n^t$  for  $t > 0$ .

**2.9.** It follows from the perfect pairing above that  $\dim S_n^N/T_{1,N} = \dim R_F^N/T'_0 = 1$ . Using the definition of  $T_1$ , it follows,

$$S_n^k/T_{1,k} \times S_n^{N-k}/T_{1,N-k} \rightarrow S_n^N/T_{1,N}$$

is a perfect pairing. Hence,  $\dim S_n^d/T_{1,d} = \dim S_n^{N-d}/T_{1,N-d}$ .

**Lemma 2.10.** The tangent space  $T_X(\text{NL}_{d,n}(\gamma))$  coincides with  $T_{1,d}$ .

*Proof.* Note that  $H \in T_{1,d}$  if and only if  $\bar{H} \otimes R_F^{N-d}$  is contained in  $T'_0$  which by definition is equivalent to  $\bar{P}\bar{H} \otimes R_F^{N-d} = 0$  in  $R_F^{2N}$ . Using the perfect pairing 2.8 we can conclude that  $\bar{P}\bar{H} = 0$  in  $R_F^{N+d}$ . This is equivalent to  $H \in T_X(\text{NL}_{d,n}(\gamma))$ .  $\square$

**2.11.** Suppose that  $Z$  is defined by  $n+1$  polynomials  $P_0, \dots, P_n$ . Since  $Z \subset X$ , we can assume that there exist polynomials  $Q_0, \dots, Q_n$  of degree  $d - \deg P_i$ , respectively such that  $X$  is defined by a polynomial of the form  $P_0Q_0 + \dots + P_nQ_n$ . Let  $I$  be the ideal in  $S_n$  generated by  $P_0, \dots, P_n$  and  $Q_0, \dots, Q_n$ .

**Proposition 2.12.** The  $k$ -graded pieces,  $T_{1,k} = I_k$  for all  $k \leq N$ .

*Proof.* Denote by  $Z_1$  the subschemes in  $\mathbb{P}^{2n+1}$ , defined by  $Q_0 = P_1 = \dots = P_n = 0$ . Since  $Z \cup Z_1$  is the intersection of  $X$  and  $\{P_1 = \dots = P_n = 0\}$ , then  $[Z] = -[Z_1] \mod \mathbb{Q}H_X^n$  in the cohomology group  $H^{n,n}(X, \mathbb{Q})$ . So,  $[Z]_{\text{prim}} = -[Z_1]_{\text{prim}}$ . Denote by  $Z_2$  the subvariety defined by  $Q_0 = \dots = Q_n = 0$ . Proceeding similarly, we get  $[Z]_{\text{prim}} = a[Z_2]_{\text{prim}}$  for some integer  $a$ . Using [GH83, 4.a.4], we have  $(P_0, \dots, P_n, Q_0, \dots, Q_n) \subset T_1$ . Since  $X$  is smooth the sequence  $\{P_0, \dots, P_n, Q_0, \dots, Q_n\}$  is a regular sequence. Using Theorem 2.5 we can conclude that  $\dim S_n^N/I_N = 1$ , where  $I_N$  denotes the degree  $N$  graded piece of  $I$  and

$$S/I|_k \times S/I|_{N-k} \rightarrow S/I|_N$$

is perfect pairing. So,  $I$  is Gorenstein of socle degree  $N$  contained in  $T_1$  which is Gorenstein of the same socle degree. So,  $T_{1,k} = I_k$  for all  $k \leq N$ .  $\square$

**2.13.** The parameter space, say  $H$  of complete intersection subschemes in  $\mathbb{P}^{2n+1}$  of codimension  $n+1$ , defined by  $n+1$  polynomials of degree  $\deg(P_i)$ , respectively is irreducible. In particular, it is an open subscheme of

$$\mathbb{P}(S_n^{\deg P_0}) \times \dots \times \mathbb{P}(S_n^{\deg P_n})$$

which is irreducible. Denote by  $R_0$  the Hilbert polynomial of  $Z$  as a subscheme in  $\mathbb{P}^{2n+1}$ . Consider the flag Hilbert scheme  $H_{R_0, Q_d}$  and the projection map  $\text{pr}_1$  which is the projection onto the first component. Since the generic fiber of  $\text{pr}_1$  is isomorphic to  $\mathbb{P}(I_d(Z))$  for the generic subscheme  $Z$  on  $\text{pr}_1 H_{R_0, Q_d}$ , it is irreducible, where  $I_d(Z)$  is the degree  $d$  graded piece of the ideal,  $I(Z)$ , of  $Z$ . So, there exists a unique irreducible component in  $H_{R_0, Q_d}$  such that the image under  $\text{pr}_1$  of this component coincides with  $H$ . For simplicity of notation, we denote by  $H_{R_0, Q_d}$  this irreducible component, since we are interested only in this scheme.

**2.14** (Proof of Theorem 1.1). Using basic deformation theory and Hodge theory, we can conclude that  $\text{pr}_2(H_{R_0, Q_d})$  is contained in  $\overline{\text{NL}_{d,n}(\gamma)}$ . So,

$$\text{codim } \text{pr}_2(H_{R_0, Q_d}) \geq \text{codim } \text{NL}_{d,n}(\gamma) \geq \text{codim } T_X \text{NL}_{d,n}(\gamma).$$

Now, there is a natural morphism, denoted  $p$  from  $T_{1,d}$  to  $H_{Q_d}$  which maps  $F_1$  to the zero locus of  $F_1$ . Since every element of  $T_{1,d}$  defines a hypersurface containing a subscheme with Hilbert polynomial  $R_0$ ,  $\overline{\text{pr}_2(H_{R_0, Q_d})}$  contains  $\overline{\text{Im } p}$ . Since the zero locus of a polynomial is invariant under multiplication by a scalar,

$$\dim T_{1,d} = \dim \overline{\text{Im } p} + 1.$$

Finally,

$$\begin{aligned} \text{codim } \text{pr}_2(H_{R_0, Q_d}) &= \dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d))) - \dim \overline{\text{pr}_2(H_{R_0, Q_d})} \leq \\ &\leq (h^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d)) - 1) - \dim \overline{\text{Im } p} \leq h^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d)) - \dim T_{1,d} = \text{codim } T_X \text{NL}_{d,n}(\gamma) \end{aligned}$$

where the last equality follows from Lemma 2.10. This proves Theorem 1.1.

**2.15.** Furthermore, note that  $\overline{\text{NL}_{d,1}(\gamma)}$  is reduced and parametrizes all degree  $d$  surfaces in  $\mathbb{P}^3$  containing a complete intersection curve with the Hilbert polynomial  $P$ . This is a part of Theorem 1.2(ii).

### 3 Proof of Theorem 1.2

**3.1.** If  $C_1$  is a complete intersection curve then reducedness of  $\text{NL}_{d,1}([C_1])$  follows from 2.15.

If  $C_1$  is an integral curve,  $\deg(C_1) < 4$  and  $C_1$  not complete intersection then  $C_1$  is a twisted cubic. Recall, the twisted cubic  $C_1$  is generated by 3 polynomials of degree 2 each. Suppose  $P$  is a polynomial in  $T_1$  such that  $P$  is not contained in  $I(C_1)$ . Since the zero locus of the ideal, say  $I'$  generated by  $I(C_1)$  and  $P$  is non-empty,  $J_F$  (which is base point free) is not contained in  $I'$ . Since the Jacobian ideal,  $J_F \subset T_1$ , we can show that there is a regular sequence in  $T_1$  consisting of 4 elements, two of which are the generators of  $I(C_1)$ , the third one is  $P$  and the forth is an element in  $J_F^{d-1}$ . Since,  $\text{codim } T_{1,2d-4} = 1$ , Theorem 2.5 implies  $2+2+\deg(P)+d-1-4 \geq 2d-4$ . So,  $\deg(P) \geq d-3$ . By Proposition 2.12,

$$\begin{aligned} \text{codim } T_X(\text{NL}_{d,1}(\gamma)) &= \text{codim } T_{1,d} = \text{codim } T_{1,d-4} = \\ &= \text{codim } I_{d-4}(C_1) = 3.(d-4) + 1 = 3d-11. \end{aligned}$$

**3.2.** Using basic deformation theory and Hodge theory we can conclude that there exists an unique irreducible component  $H$  of  $H_{P,Q_d}$  whose generic element is  $(C, X)$ , where  $C$  is a twisted cubic contained in  $X$  such that  $\text{pr}_2(H)$  is contained in  $\overline{\text{NL}_{d,1}(\gamma)}$ . It is easy to compute that  $\text{codim}(\text{pr}_2(H)) = 3d-11$ . So,

$$3d-11 \geq \text{codim } \overline{\text{NL}_{d,1}(\gamma)} \geq \text{codim } T_X(\text{NL}_{d,1}(\gamma)) = 3d-11.$$

So,  $\overline{\text{NL}_{d,1}(\gamma)}$  is reduced and parametrizes smooth degree  $d$  surfaces containing a twisted cubic. This finishes the proof of (i).

**3.3.** We now recall a result due to Otwinowska that will help us make the characterization of

the irreducible components of  $\text{NL}_{d,1}$  as in Theorem 1.2(ii).

**Theorem 3.4** ([Otw04, Theorem 1]). Let  $\gamma$  be a Hodge class on a smooth degree  $d$  surface. There exists  $C \in \mathbb{R}_+^*$  depending only on  $r$  such that for  $d \geq C(r-1)^8$  if  $\text{codim NL}_{d,1}(\gamma) \leq (r-1)d$  then  $\gamma_{\text{prim}} = \sum_{i=1}^t a_i [C_i]_{\text{prim}}$  where  $a_i \in \mathbb{Q}^*$ ,  $C_i$  are integral curves and  $\deg(C_i) \leq (r-1)$  for  $i = 1, \dots, t$  for some positive integer  $t$ .

This implies the following:

**Proposition 3.5.** Let  $d \gg 0$ ,  $\gamma$  be a Hodge class in a smooth degree  $d$  surface in  $\mathbb{P}^3$  such that  $\text{codim } \overline{\text{NL}}_{d,1}(\gamma) \leq 3d$ . Then there exists integral curves  $C_1, \dots, C_t$  of degree at most 3 such that  $\gamma = \sum_{i=1}^t a_i [C_i] + bH_X$  for some integers  $a_i, b$  and  $\text{NL}(\gamma)_{\text{red}}$  is the same as  $\bigcap_{i=1}^t \text{NL}([C_i])_{\text{red}}$ .

*Proof.* Let  $X \in \text{NL}_{d,1}(\gamma)$ . There exists a maximal  $\mathbb{Q}$ -vector space  $\Lambda \subset H^2(X, \mathbb{Q})$  such that  $\Lambda$  remains of type  $(1, 1)$  in  $\text{NL}_{d,1}(\gamma)$  i.e.,  $\text{NL}_{d,1}(\gamma)_{\text{red}} = \bigcap_{\gamma_i \in \Lambda} \text{NL}_{d,1}(\gamma_i)_{\text{red}}$ . There exists a surface  $X' \in \text{NL}_{d,1}(\gamma)$  such that the Néron-Severi group  $\text{NS}(X')$  is the translate (under deformation from  $X$  to  $X'$ ) of  $\Lambda$  in  $H^2(X', \mathbb{Z})$  which we again denote by  $\Lambda$  for convinience. Then, Theorem 3.4 implies that any  $\gamma \in \Lambda$  is of the form  $\sum_i a_i [C_i] + bH_X$  with  $\deg(C_i) \leq 3$ . So,  $\Lambda$  is generated by classes of curves of degree at most 3 and  $H_X$ . Note that the classes of these curves are also contained in  $\Lambda$  since  $\Lambda$  is the complete Néron-Severi group of  $X'$ . This proves the proposition, which is also the first part of Theorem 1.2(ii).  $\square$

**3.6.** We now come to the proof of the final part of the theorem. Suppose now that  $\gamma$  is as in the above proposition i.e., of the form  $\sum_{i=1}^t a_i [C_i] + bH_X$  such that  $\text{NL}(\gamma)_{\text{red}} = \bigcap_{i=1}^t \text{NL}([C_i])_{\text{red}}$ . Denote by  $\bar{P}_i$  the element in  $R_F^{2d-4}$  such that  $\alpha_2(\bar{P}_i) = [C_i]_{\text{prim}}$  for  $i = 1, \dots, t$ . Since  $\alpha_2$  is a linear map,  $\alpha_2(\sum_{i=1}^r a_i \bar{P}_i) = \sum_i a_i [C_i]_{\text{prim}}$ . Denote by  $\bar{P} := \sum_{i=1}^r a_i \bar{P}_i$ . So,  $\alpha_2(\bar{P}) = \gamma$ . Denote by  $T_{1,d-4}^{[C_i]}$  the corresponding  $T_{1,d-4}$  in 2.8 obtained by replacing  $P$  by  $\bar{P}_i$  for  $i = 1, \dots, r$ . Note that  $\text{codim } T_X \text{NL}_{d,1}(\gamma) = \text{codim } T_{1,d} = \text{codim } T_{1,d-4}$ , where the last equality is due to perfect pairing. Note that,  $\bigcap_{i=1}^r T_{1,d-4}^{[C_i]} \subset T_{1,d-4}$  because  $\bar{P} = \sum_i a_i \bar{P}_i$ , so  $\bigcap_{i=1}^r \ker \bar{P}_i \subset \ker \bar{P}$ . Therefore,  $\text{codim } T_X \text{NL}_{d,1}(\gamma) \leq \text{codim } I_{d-4}(\bigcup_{i=1}^r C_i)$ .

Before we go to the last step of the proof we need the following computation:



**Lemma 3.7.** Let  $d \geq 5$  and  $C$  be an effective divisor on a smooth degree  $d$  surface  $X$  of the form  $\sum_i a_i C_i$ , where  $C_i$  are integral curves with  $\deg(C) + 4 \leq d$ . Then,  $\dim |C| = 0$ , where  $|C|$  is the linear system associated to  $C$ .

*Proof.* Let  $C = \sum_i a_i C_i$  with  $C_i$  integral. Then,

$$\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = a_i C_i^2 + \sum_{j \neq i} a_j C_i \cdot C_j.$$

Denote by  $e_i := \deg(C_i)$ . Using the adjunction formula and the fact that  $K_X \cong \mathcal{O}_X(d-4)$ , we have that

$$\begin{aligned} \deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) &= 2a_i \rho_a(C_i) - 2a_i - (d-4)a_i e_i + \sum_{j \neq i} a_j C_i \cdot C_j \\ &\leq a_i(e_i^2 - (d-1)e_i) + \sum_{j \neq i} a_j C_i C_j \\ &\leq a_i(e_i^2 - 3e_i - e_i \sum_j a_j e_j) + \sum_{j \neq i} a_j e_i e_j. \end{aligned}$$

The first inequality follows from the bound on the genus of a curve in  $\mathbb{P}^3$  in terms of its degree (see [Har77, Example 6.4.2]). The second inequality follows from the facts that  $d \geq \deg(C) + 2$  and  $C_i \cdot C_j \leq e_i e_j$ . It then follows directly that  $\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) < 0$ . This implies that  $h^0(C_i, (\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = 0$  for all  $i$ . This implies that  $h^0(C, \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C) = 0$ . Since  $h^1(\mathcal{O}_X) = 0$  (by Lefschetz hyperplane section Theorem) and  $h^0(\mathcal{O}_X) = 1$ , using the long exact sequence associated to the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C \rightarrow 0 \quad (1)$$

we get that  $h^0(\mathcal{O}_X(C)) = 1$ . Since  $|C| = \mathbb{P}(H^0(\mathcal{O}_X(C)))$ , the lemma follows.  $\square$

**3.8** (Proof of Theorem 1.2). Using Proposition 2.12 and 3.1,  $\bigcap_{i=1}^r T_{1,d-4}^{[C_i]} = I_{d-4}(\bigcup_{i=1}^r C_i)$  is contained in  $T_X \text{NL}_{d,1}(\gamma)$ . Denote by  $P_i$  the Hilbert polynomial of  $C_i$  for  $i = 1, \dots, t$ . By Theorem 1.2(i), there exists an irreducible component of  $H_{P_i, Q_d}$  such that its image under the natural projection morphism  $\text{pr}_2$  (onto the second component) is isomorphic to  $\overline{\text{NL}_{d,1}([C_i])}_{\text{red}}$ . So, there exists an irreducible component, say  $H_\gamma$  of  $H_{P_1, Q_d} \times_{H_{Q_d}} \dots \times_{H_{Q_d}} H_{P_t, Q_d}$  such that

$\text{pr}_2(H_\gamma)_{\text{red}} = \overline{\text{NL}(\gamma)_{\text{red}}}$ , where  $\text{pr}_2$  is the natural morphism from  $H_\gamma$  to  $H_{Q_d}$ . Denote by  $L_\gamma := \text{pr}(H_\gamma)$ , where  $\text{pr}$  is the natural projection morphism to  $H_{P_1} \times \dots \times H_{P_t}$ . A generic  $t$ -tuple of curves  $(C_1, \dots, C_t) \in H_{P_1} \times \dots \times H_{P_t}$  does not intersect each other. Since there exists  $i, j, i \neq j$  such that  $C_i \cap C_j \neq \emptyset$ , we have  $\dim L_\gamma < \sum_{i=1}^t \dim H_{P_i}$ . Lemma 3.7 implies that  $\dim |C_i| = 0$  for  $i = 1, \dots, t$ . It is then easy to see that  $\text{codim NL}_{d,1}(\gamma) = \text{codim } I_d(\bigcup_{i=1}^t C_i) - \dim L_\gamma$ . If  $\text{codim } I_{d-4}(\bigcup_{i=1}^t C_i) \stackrel{\dagger}{\leq} \text{codim } I_d(\bigcup_{i=1}^t C_i) - \sum_{i=1}^t \dim H_{P_i}$  then

$$\begin{aligned} \text{codim } T_X \text{NL}_{d,1}(\gamma) &\leq \text{codim } I_{d-4}(\bigcup_{i=1}^t C_i) \stackrel{\dagger}{\leq} \text{codim } I_d(\bigcup_{i=1}^t C_i) - \sum_{i=1}^t \dim H_{P_i} \\ &< \text{codim } I_d(\bigcup_{i=1}^t C_i) - \dim L_\gamma = \text{codim NL}_d(\gamma), \end{aligned}$$

where the first inequality follows from 3.6. Since  $d \gg 0$ , using the Hilbert polynomial of  $\bigcup C_i$ , the inequality  $\dagger$  is equivalent to  $\sum_{i=1}^t \dim H_{P_i} \leq 4 \sum_{i=1}^t \deg(C_i)$ . Since  $\deg(C_i) < 4$  and  $C_i$  is integral, it is easy to compute that  $\dim H_{P_i}$  is infact equal to  $4 \deg(C_i)$ . This proves (ii). Hence, completes the proof of the theorem.

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